Financial Applications of the Zeta Process

Lane P. Hughston

Department of Mathematics, Imperial College London
London SW7 2AZ, United Kingdom
lane.hughston@imperial.ac.uk

Asymptotic Statistics, Risk and Computation in Finance and Insurance
Tokyo Institute of Technology, 14 December 2010

Collaborators: D. C. Brody, S. Lyons, M. R. Pistorius

Introduction

The zeta distribution, sometimes also called the Zipf distribution, is the discrete analogue of the so-called Pareto distribution, and has been used to model a variety of interesting phenomena with fat-tailed power-law behaviour.

Examples include word frequency, corporate income, citations of scientific papers, web hits, copies of books sold, frequency of telephone calls, magnitudes of earthquakes, diameters of moon craters, intensities of solar flares, intensities of wars, personal wealth, frequencies of family names, frequencies of given names, populations of cities.

It makes sense therefore to consider financial contracts for which the payoff is represented by a random variable of this type.

This talk will present an overview of some of the properties of the zeta distribution and the associated multipicative Lévy process, which we shall call the zeta process, with a view to financial applications.

The material under consideration can be regarded more generally as part of an ongoing program, being pursued by a number of authors, devoted to various aspects of the relationship between probability and number theory.

Definitions

We fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and let Z be a random variable taking values in the positive integers, with

$$\mathbb{P}(Z = n) = \frac{n^{-s}}{\zeta(s)}. \tag{1}$$

Here $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, for some fixed $s > 1$.

We say that Z is zeta-distributed with parameter s, or simply that Z has a Zeta(s) distribution. Sometimes we write $\mathbb{P}^s$ instead of $\mathbb{P}$ to emphasize the choice of parameters.

What is so special about the zeta distribution?

One way to motivate the introduction of the zeta distribution is through a maximum entropy argument.

For this purpose it will be useful to consider first the geometric distribution.

Let us write $\{q_n\}_{n\in\mathbb{N}_0}$ for a probability distribution on $\mathbb{N}_0$, and

$$S = -\sum_{n=0}^{\infty} q_n \ln q_n \tag{2}$$

for the associated Shannon entropy.

Let the random variable A have the distribution $\{q_n\}$, and choose $\{q_n\}$ so that the entropy S is maximal under the constraint that $\mathbb{E}[A]$ has a specified value.

Then one can show that A has a geometric distribution

$$\mathbb{P}(A = n) = q^n(1 - q), \tag{3}$$

where the parameter $q > 0$ is determined by the mean $\mathbb{E}[A] = q/(1 - q)$.

Now, let Z take values in $\mathbb{N}$, with the distribution $\{q_n\}_{n\in\mathbb{N}}$, and choose $\{q_n\}_{n\in\mathbb{N}}$ such that S is maximized for some fixed value of $\mathbb{E}[\ln Z]$.

Then Z has a zeta distribution with parameter s, where s is determined for a given value of the mean of the logarithm of Z by the relation

$$\mathbb{E}[\ln Z] = -\frac{\zeta'(s)}{\zeta(s)}. \tag{4}$$

Interpretation

Let $\{Z_i\}_{i=1\ldots k}$ represent a collection of independent outcomes of a sample from the zeta distribution. Then by the strong law of large numbers, for large k we have

$$\mathbb{E}[\ln Z] \approx \ln \Big(\prod_{i=1}^{k} Z_i \Big)^{1/k}. \tag{5}$$

Thus, intuitively, fixing $\mathbb{E}[\ln Z]$ means fixing the geometric mean of Z.

It follows that the zeta distribution is the maximum entropy distribution for a given value of the geometric mean.

The fact that it is the geometric mean that arises in the present context reminds us that the zeta function is associated with *multiplicative* properties of the natural numbers.

Divisibility property

Let us write $m|Z$ for the event $\{\ m$ divides $Z\ \}$.

Then one can show that Z has the following so-called "factorization" property:

Proposition 1. *Let Z have a zeta distribution. Then for m and n relatively prime we have*

$$\mathbb{P}[m|Z \;\&\; n|Z] = \mathbb{P}[m|Z]\;\mathbb{P}[n|Z]. \tag{6}$$

Proof. First, we note that

$$\mathbb{P}[m|Z] = \sum_{k=1}^{\infty} \mathbb{P}[Z = km] \tag{7}$$

$$= \sum_{k=1}^{\infty} \frac{(km)^{-s}}{\zeta(s)} \tag{8}$$

$$= m^{-s}. \tag{9}$$

Likewise, we have

$$\mathbb{P}[mn|Z] = (mn)^{-s}. \tag{10}$$

But, if $(m, n) = 1$, then $\{mn|Z\} = \{m|Z \;\&\; n|Z\}$, and the result follows. □

Remark 1. *We observe that $\mathbb{P}\{m$ does not divide $Z\}$ is given for any m by*

$$\mathbb{P}(m \nmid Z) = 1 - m^{-s}. \tag{11}$$

Thus the relation

$$\mathbb{P}(Z = 1) = \prod_{p \in \mathcal{P}} \mathbb{P}(p \nmid Z) \tag{12}$$

gives us a probabilistic interpretation of the Euler formula,

$$\frac{1}{\zeta(s)} = \prod_{p \in \mathcal{P}} (1 - p^{-s}). \tag{13}$$

Change of measure

It will be useful in what follows to have at our disposal expressions for the moment generating function and the characteristic function of $\ln Z$.

In particular, we have:

$$\mathbb{E}[e^{-i\lambda \ln Z}] = \mathbb{E}[Z^{-i\lambda}] \tag{14}$$

$$= \sum_{n=1}^{\infty} \frac{n^{-s-i\lambda}}{\zeta(s)} \tag{15}$$

$$= \frac{\zeta(s + i\lambda)}{\zeta(s)}. \tag{16}$$

Similarly for $\lambda > 1 - s$ we have:

$$\mathbb{E}[Z^{-\lambda}] = \frac{\zeta(s+\lambda)}{\zeta(s)}. \tag{17}$$

We already know that the mean of $\ln Z$ is given by $-\zeta'(s)/\zeta(s)$, and from the above one can deduce that

$$\mathrm{Var}(\ln Z) = \frac{\zeta''(s)}{\zeta(s)} - \frac{(\zeta'(s))^2}{(\zeta(s))^2} \tag{18}$$

It follows that

$$(\ln \zeta(s))'' > 0. \tag{19}$$

In other words, $\zeta(s)$ is log-convex.

What is "log-convexity"?

This is the multiplicative analogue of ordinary convexity.

For example, if a, b, c and d denote four sequential real numbers each greater than one, then we have

$$\frac{\zeta(a)\zeta(d)}{\zeta(b)\zeta(c)} > 1. \tag{20}$$

Suppose now that $s > 1$ and that Z has a zeta distribution (with parameter s) under $\mathbb{P}^s$. Let $\lambda > 1 - s$.

Then of course $\mathbb{P}^s[\omega \in C] = \mathbb{E}^s[\mathbf{1}_C]$ for any $C \in \mathscr{F}$. We define a new measure $\mathbb{P}^{s+\lambda}$ on $(\Omega, \mathscr{F})$ by setting

$$\mathbb{P}^{s+\lambda}[\omega \in C] = \mathbb{E}^s\left[\mathbf{1}_C \frac{Z^{-\lambda}}{\mathbb{E}^s[Z^{-\lambda}]}\right]. \tag{21}$$

Then under $\mathbb{P}^{s+\lambda}$ we have

$$\mathbb{P}^{s+\lambda}[Z = n] = \mathbb{E}^s\left[\mathbf{1}_{\{Z=n\}} \frac{Z^{-\lambda}}{\mathbb{E}^s[Z^{-\lambda}]}\right] \tag{22}$$

$$= \frac{n^{-\lambda}\mathbb{P}^s[Z = n]}{\mathbb{E}^s[Z^{-\lambda}]} \tag{23}$$

$$= n^{-\lambda}\frac{n^{-s}}{\zeta(s)}\frac{1}{\mathbb{E}^s[Z^{-\lambda}]}. \tag{24}$$

But $\mathbb{E}^s[Z^{-\lambda}] = \zeta(s+\lambda)/\zeta(s)$, and it follows that

$$\mathbb{P}^{s+\lambda}[Z = n] = \frac{n^{-(s+\lambda)}}{\zeta(s+\lambda)}. \tag{25}$$

Prime-factor representation

To proceed further it will be useful to decompose Z into prime factors. For each $\omega \in \Omega$ we have an expression of the form

$$Z = p_1^{A_1} p_2^{A_2} p_3^{A_3} \cdots . \tag{26}$$

In this way we define the random variables $A_k(\omega)$ for $k = 1, 2, 3, \ldots$, taking values in $\mathbb{N}_0$.

Proposition 2. *Z has a zeta distribution with parameter s if and only if the A_k's are independent geometric random variables with*

$$\mathbb{P}[A_k = \alpha_k] = q_k^{\alpha_k}(1 - q_k), \tag{27}$$

where $q_k = p_k^{-s}$.

Proof. On the one hand, if the A_k's are as given we have

$$\mathbb{P}[Z = n] = \prod_k \mathbb{P}[A_k = \alpha_k], \tag{28}$$

where $n = \prod_k p_k^{\alpha_k}$.

It follows that

$$\mathbb{P}[Z=n] = \prod_k q_k^{\alpha_k}(1-q_k) \tag{29}$$

$$= \prod_k p_k^{-s\alpha_k}(1-p_k^{-s}) \tag{30}$$

$$= (\prod_k p_k^{\alpha_k})^{-s} \prod_k (1-p_k^{-s}) \tag{31}$$

$$= \frac{n^{-s}}{\zeta(s)}, \tag{32}$$

as required. On the other hand, suppose Z has a zeta distribution with parameter s. Then for each k we have

$$\mathbb{P}[A_k=\alpha_k] = \mathbb{P}[p_k^{\alpha_k}|Z \;\&\; p_k^{\alpha_k+1} \nmid Z] \tag{33}$$

$$= \mathbb{P}[p_k^{\alpha_k}|Z] - \mathbb{P}[p_k^{\alpha_k+1}|Z] \tag{34}$$

$$= p_k^{-s\alpha_k} - p_k^{-s(\alpha_k+1)} \tag{35}$$

$$= (p_k^{-s})^{\alpha_k}(1-p_k^{-s}), \tag{36}$$

as required. The independence of the events $\{p_k^{\alpha_k}|Z\}$ for different values of k then implies that the A_k's are independent. □

The prime-factor representation for Z allows us to see various properties of Z rather directly.

For example, the independence of the events $\{m|X\}$ and $\{n|X\}$ if $(m, n) = 1$ follows immediately from the independence of the A_k.

Poisson representation

The decomposition of Z into random prime factors can be pursued further to give other useful representations of the zeta distribution.

To this end we examine the characteristic function of $\ln Z$.

Note that $\ln Z = \sum_k A_k \ln p_k$. We thus have:

$$\mathbb{E}[e^{-i\lambda \ln Z}] = \mathbb{E}[e^{-i\lambda \sum_k A_k \ln p_k}] \tag{37}$$

$$= \prod_k \mathbb{E}[e^{-i\lambda A_k \ln p_k}]. \tag{38}$$

Now, for each k we can work out the corresponding term in the product above.

For a typical term we have (suppressing k):

$$\mathbb{E}[e^{-i\lambda A\ln p}] = \sum_{n=0}^{\infty} e^{-i\lambda n\ln p}p^{-sn}(1-p^{-s}) \tag{39}$$

$$= \sum_{n=0}^{\infty}(p^{-(s+i\lambda)})^n(1-p^{-s}) \tag{40}$$

$$= \frac{1-p^{-s}}{1-p^{-(s+i\lambda)}}. \tag{41}$$

For each p we have:

$$\ln\frac{1-p^{-s}}{1-p^{-(s+i\lambda)}} = \ln(1-p^{-s}) - \ln(1-p^{-(s+i\lambda)}) \tag{42}$$

$$= -\sum_{n=1}^{\infty}\frac{1}{n}p^{-sn} + \sum_{n=1}^{\infty}\frac{1}{n}p^{-(s+i\lambda)n} \tag{43}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}p^{-sn}[p^{-i\lambda n}-1] \tag{44}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}p^{-sn}\left[e^{-i\lambda n\ln p}-1\right]. \tag{45}$$

Therefore, by exponentiation we have:

$$\mathbb{E}[e^{-i\lambda A\ln p}] = \frac{1-p^{-s}}{1-p^{-(s+i\lambda)}} = \prod_{n=1}^{\infty} \exp\left[\frac{1}{n}p^{-sn}\left(e^{-i\lambda n\ln p}-1\right)\right]. \tag{46}$$

If N is a Poisson random variable taking values in the lattice $\{m\Delta\}$, $m \in \mathbb{N}_0$, $\Delta \in \mathbb{R}^+$, with intensity Λ, we have

$$\mathbb{P}(N = m\Delta) = \frac{\Lambda^m}{m!}e^{-\Lambda}, \tag{47}$$

for $m = 0, 1, 2, \ldots$. Its characteristic function is

$$\mathbb{E}[e^{-i\lambda N}] = \exp[\Lambda(e^{-i\lambda\Delta} - 1)]. \tag{48}$$

Therefore the A_k's can be represented (in law) in the form

$$A_k = \sum_{n=1}^{\infty} n\pi_{kn}. \tag{49}$$

Here the π_{kn}'s are independent Poisson random variables, with intensity

$$\mathbb{E}[\pi_{kn}] = \frac{1}{n}p_k^{-sn}. \tag{50}$$

Thus a zeta-distributed random variable has a multiplicative Poisson representation, given by:

$$Z = \prod_{k,n} p_k^{\, n\pi_{nk}}. \tag{51}$$

Lévy processes

The fact that the random variable $\ln Z$ has an infinitely divisible distribution means that there is a natural Lévy process $\{X_t\}_{t\geq 0}$ associated with it.

In particular we have

$$\mathbb{E}[e^{-\lambda X_t}] = e^{t\psi(\lambda)}, \tag{52}$$

where

$$\psi(\lambda) = \ln[e^{-\lambda X_1}] \tag{53}$$

$$= \ln \frac{\zeta(s+\lambda)}{\zeta(s)}. \tag{54}$$

It follows that

$$\mathbb{E}[e^{-\lambda X_t}] = \mathbb{E}[Z_t^{-\lambda}] = \left(\frac{\zeta(s+\lambda)}{\zeta(s)}\right)^t . \tag{55}$$

Here we write

$$Z_t = \exp(X_t) \tag{56}$$

for the exponential process associated with $\{X_t\}_{t\geq 0}$.

By use of the prime decomposition of $\{Z_t\}$ we infer the existence of a set of independent Lévy processes $\{A_{kt}\}_{t\geq 0}$ such that

$$Z_t = \prod_k p_k^{A_{kt}}. \tag{57}$$

It follows then that

$$\mathbb{E}[e^{-\lambda A_{kt} \ln p_k}] = \left(\frac{1-p_k^{-s}}{1-p_k^{-(s+\lambda)}}\right)^t . \tag{58}$$

This shows that for each t the random variable A_{kt} has a negative-binomial distribution.

Alternatively, we can write

$$Z_t = \prod_{k,n} p_k^{n\pi_{k,n}(t)}. \tag{59}$$

Here $\{\pi_{k,n}(t)\}_{t\geq 0}$ is a collection of independent Poisson processes, with

$$\mathbb{E}[\pi_{k,n}(t)] = tn^{-1}p_k^{-sn}. \tag{60}$$

Distribution of zeta process

We observe that $0 < x < 1$ and real $t > 0$ we have

$$\frac{1}{(1-x)^t} = \sum_{m=0}^{\infty} \omega_m(t)\, x^m \tag{61}$$

where

$$\omega_m(t) = \frac{\Gamma(m+t)}{\Gamma(t)\Gamma(m+1)}. \tag{62}$$

Thus if we write

$$q_m = \omega_m(t)\, x^m (1-x)^t, \tag{63}$$

we see that $\{q_m\}$ is a probability distribution.

Then if we set $x = p_k^{-s}$ we obtain

$$\mathbb{P}(A_{kt} = \alpha_k) = \omega_{\alpha_k}(t)\, p_k^{-s\alpha_k}(1 - p_k^{-s})^t. \tag{64}$$

Here

$$\omega_{\alpha_k}(t) = \frac{\Gamma(\alpha_k + t)}{\Gamma(t)\Gamma(\alpha_k + 1)}. \tag{65}$$

Thus the distribution of Z_t is given by

$$\mathbb{P}[Z_t = n] = \prod_k \mathbb{P}[A_{kt} = \alpha_k] \tag{66}$$

$$= \prod_{k:\ p_k^{\alpha_k}||n} \omega_{\alpha_k}(t)\, p_k^{-s\alpha_k} \prod_k (1 - p_k^{-s})^t. \tag{67}$$

Thus

$$\mathbb{P}[Z_t = n] = \frac{c_n(t) n^{-s}}{(\zeta(s))^t}, \tag{68}$$

where

$$c_n(t) = \prod_{k:\ p_k^{\alpha_k}||n} \frac{\Gamma(\alpha_k + t)}{\Gamma(t)\Gamma(\alpha_k + 1)}. \tag{69}$$

It is interesting to observe that this gives us the following Dirichlet expansion:

$$(\zeta(s))^t = \sum_{n=0}^{\infty} c_n(t) n^{-s}. \tag{70}$$

Martingales

A variety of martingales are associated with the zeta function.

In the case of $X_t = \ln Z_t$ we find that

$$\left\{X_t + t\frac{\zeta'(s)}{\zeta(s)}\right\}_{t\geq 0} \tag{71}$$

is a $\mathbb{P}^s$-martingale.

Likewise for each $\lambda > 1 - s$ the process $\{\rho_t\}_{t\geq 0}$ defined by

$$\rho_t = \left(\frac{\zeta(s)}{\zeta(s+\lambda)}\right)^t Z_t^{-\lambda} \tag{72}$$

is a $\mathbb{P}^s$-martingale that can be used as the basis for a measure change.

Asset pricing

We proceed therefore as follows, taking a pricing kernel approach.

Let us fix $\mathbb{P}^s$ as the "real-world" measure, and let $\{Z_t\}$ be the associated zeta process.

Then assuming a constant interest rate r, and requiring that $\lambda > 0$, we can use the martingale $\{\rho_t\}$ to construct a pricing kernel $\{\pi_t\}$ by setting

$$\pi_t = e^{-rt}\rho_t. \tag{73}$$

The corresponding model for an asset process is given by

$$S_t = S_0 e^{rt} \left(\frac{\zeta(s+\lambda)}{\zeta(s+\lambda-\nu)} \right)^t {Z_t}^{\nu}, \tag{74}$$

where the "volatility" ν is a parameter satisfying $0 < \nu < s - 1$.

Then on the one hand we have

$$\pi_t S_t = S_0 \left(\frac{\zeta(s)}{\zeta(s+\lambda-\nu)} \right)^t {Z_t}^{\nu-\lambda}, \tag{75}$$

from which it follows that $\{\pi_t S_t\}$ is a $\mathbb{P}^s$-martingale, as required.

On the other hand, we also have

$$\mathbb{E}[S_t] = S_0 e^{rt} \left(\frac{\zeta(s-\nu)\zeta(s+\lambda)}{\zeta(s)\zeta(s+\lambda-\nu)} \right)^t . \tag{76}$$

We therefore able to deduce that $\mathbb{E}[S_t] > S_0 e^{rt}$ for $t > 0$.

The fact that model has positive excess returns (above the interest rate) follows from the log-convexity of the Riemann zeta function. Specifically, we observe that $s - \nu < s$ and $s - \nu < s + \lambda - \nu$ and $s < s + \lambda$ and $s + \lambda - \nu < s + \lambda$.

One thus sees that it is possible to construct a consistent arbitrage-free theory of financial claims with Zipfian payoffs and zeta-like dynamics, enjoying some of the same mathematical tractability as the geometric Brownian motion model. In particular, option prices can be computed.

Despite the various artificialities involved, and the somewhat stylized nature of the distributions, this is encouraging, since it points towards the possibility of a general theory of asset pricing incorporating such phenomena.